Persistent correlations in the motion of a Brownian particle interacting with a finite-size random environment

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We study the long time evolution of a Brownian particle in a finite-size disordered system. The presence of a random static potential is shown to generate persistent correlations of kinetic energy fluctuations described by fourth-order velocity cumulant functions. We show also that the usual correlations of velocity fluctuations do not vanish at large time, provided that an external bias is applied to the sample. An appearance of persistent correlations is noted to reflect a nonergodic character of the velocity fluctuations.

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The Brownian motion in a disordered system is the subject of intense current interest [1]. The presence of a random static potential is known to model the interaction of the particle with fixed impurities or defects and to modify considerably the character of diffusion resulting in an appearance of long-range temporal velocity correlations. These anomalous phenomena can be revealed by means of the calculation of second-order velocity correlation functions. However, in some cases it is the high-order correlator that demonstrates the anomalous behavior, while the second moment of velocity fluctuations has a finite correlation time. In particular, experimental and theoretical results [2,3] devoted to the study of low-frequency $1/f$ current fluctuations in conductors reflect a similar situation.

The aim of the present paper is to investigate the long time behavior of high-order velocity correlation functions for the Brownian particle interacting with a static random field in the finite-size disordered medium. We show that in the thermodynamic equilibrium state there are persistent correlations of kinetic energy fluctuations nonvanishing at large time limit. The existence of these correlations is due to the anomalous contribution of the corresponding long time fourth-order cumulant function to the kinetic energy correlator. Here we use the weak disorder model and suppose that the random static potential has a finite correlation length. Because of these assumptions, the corrections to damping and diffusion coefficients connected with impurity scattering are negligible. Due to the nonlinearity of transport processes, long time peculiarities of kinetic energy fluctuations manifest themselves in the usual second-order velocity correlation functions, provided that an external bias is applied to the sample. It should be noted that the residual correlations level is inversely proportional to the disordered system volume L^d, where d is the dimensionality of the sample and L is its linear size. Therefore, the persistent velocity correlations can be observed in small enough samples (for example, in conducting rings, containing a large number of chaotically distributed impurities or defects). The presence of residual correlations nonvanishing at large time is responsible for the principal impossibility of measuring exactly the mean kinetic energy (in a thermodynamic equilibrium state) or the mean velocity (in a nonequilibrium state) of the classical particle interacting with a quenched disorder in a finite-size system by a time averaging procedure. A minimal error of these quantities's measurement is just proportional to the relative level of residual correlations. The effect of analyzed persistent correlations may also be detected by the direct measurement of time dependence of corresponding correlation functions.

The Brownian motion of a classical particle with a mass $m=1$ interacting with a thermal bath in the presence of a random environment is usually described by the following equation [1]:

$$\ddot{r}_j(t)=\nabla_j\{Q(\mathbf{r}(t))+\Phi(\mathbf{r}(t),t)\}+E_j\ , \tag{1}$$

where $r_j(t)$ is a particle displacement ($j=1,2,\ldots,d$) and E_j is an external bias applied to the sample. The random static potential $Q(\mathbf{r})$ modeling the interaction with the disordered medium and the thermal bath potential $\Phi(r,t)$ in a finite-size system with periodic boundary conditions may be expanded in a Fourier sum,

$$\begin{Bmatrix} Q(\mathbf{r}) \\ \Phi(\mathbf{r},t) \end{Bmatrix}=L^{-d/2}\sum \begin{Bmatrix} Q_k \\ \Phi_k(t) \end{Bmatrix}\times e^{i\mathbf{k}\cdot\mathbf{r}(t)}\ , \tag{2}$$

where a wave vector $\mathbf{k}$ is quantized in units of $2\pi/L$, L being a linear size of the system (for example, if the dimensionality of the sample $d=1$, then L is the circumference of a ring). If the thermal bath response to the particle influence is taken into account, the full variable of the thermal bath $\Phi_k^h(t)$ can be represented in the form [4]

$$\Phi_k^h(t)=\Phi_k(t)+\int dt_1\varphi_k(t,t_1)e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}\ .$$

Here

$$\varphi_k(t,t_1)=\langle(i/\hbar)[\Phi_k(t),\Phi_{-k}(t_1)]_-\rangle\eta(t-t_1)$$

is the response function (retarded Green's function) of the free thermal reservoir, $\dot{\eta}(\tau)=1$ for $\tau>0$, $\eta(\tau)=0$ for $\tau<0$. We suppose that unperturbed variables $\Phi_k(t)$ of

the dissipative system and components Q_k of the random potential are independent random variables (which may be assumed to be Gaussian) with covariance

$$\langle \Phi_k(t),\Phi_{-k}(t_1)\rangle_T = W_k(t-t_1) ,$$
$$\langle Q_k,Q_{-k}\rangle_Q = M_k ,$$

and zero mean values $\langle Q_k\rangle_Q=0$, $\langle \Phi_k(t)\rangle_T=0$. Here $\langle\ \rangle_T$ and $\langle\ \rangle_Q$ are thermal averaging and averaging over the possible environments, respectively, while $\langle\ \rangle=\langle\langle\ \rangle_Q\rangle_T$ is double averaging.

According to the formalism developed in Refs. [4,5], Eq. (1) can be written in the form of the non-Markovian Langevin equation

$$\dot{v}_j(t)+G_j[v]=\zeta_j(t)+\xi_j(t)+E_j , \tag{3}$$

where

$$G_j[v]=-L^{-d}\sum ik_j\int_0^t dt_1\left\{[W_k(t-t_1)+M_k]\frac{\delta e^{i\mathbf{k}\cdot\mathbf{r}(t)}}{\delta f_{-k}(t_1)}+\varphi_k(t,t_1)e^{i\mathbf{k}\cdot\mathbf{r}(t)}e^{-i\mathbf{k}\cdot\mathbf{r}(t_1)}\right\} , \tag{4}$$

$$\begin{Bmatrix}\xi_j(t)\\ \zeta_j(t)\end{Bmatrix}=L^{-d/2}\sum ik_j\begin{Bmatrix}Q_k\\ \Phi_k(t)\end{Bmatrix}\times e^{i\mathbf{k}\cdot\mathbf{r}(t)}-L^{-d}\sum ik_j\int_0^t dt_1\begin{Bmatrix}M_k\\ W_k(t-t_1)\end{Bmatrix}\times\frac{\delta e^{i\mathbf{k}\cdot\mathbf{r}(t)}}{\delta f_{-k}(t_1)} , \tag{5}$$

are the collision term and the random forces, respectively, the latter being due to the interaction of the Brownian particle with disordered medium $\{\xi(t)\}$ and thermal bath $\{\zeta(t)\}$; $f_k(t)$ is an auxiliary deterministic force additive with respect to potentials $\Phi_k(t)$ and Q_k. It should be noted that for Gaussian variables Q_k and $\Phi_k(t)$ the Furutsu-Novikov theorem [6] is fair; therefore, the random force $\xi_j(t)$ and $\zeta_j(t)$ has a mean zero value $\langle\xi_j(t)\rangle_Q=0$, $\langle\zeta_j(t)\rangle_T=0$. Let the dissipative system be characterized by temperature T and a small but finite time scale τ_0. Then the response function $\varphi_k(\tau)$ and the correlation function $W_k(\tau)$ in the quasiclassical limit can be approximated by the following expressions:

$$\varphi_k(\tau)=(C_k/\tau_0^2)\exp(-\tau/\tau_0)\eta(\tau) ,$$
$$W_k(\tau)=T(C_k/\tau_0)\exp(-|\tau|/\tau_0) . \tag{6}$$

If the correlation time is $\tau_0\ll(v_Tk_0)^{-1}$, where $v_T=(T/m)^{1/2}$ is the thermal speed of the Brownian particle, k_0 will be the maximum wave vector of the thermal bath, the particle relaxation being, in the main, determined by the thermostat reaction to the particle influence. Due to the finiteness of the correlation time τ_0, the collision term $G[v]$ will be the nonlinear function of the particle velocity $v(t)$,

$$G[v]\cong\gamma[1-v^2(t)/v_0^2]v(t) , \tag{7}$$

with $\gamma=L^{-1}\sum k^2C_k$, $v_0^2=\gamma\{L^{-1}\sum k^4C_k\tau_0^2\}^{-1}$. Here γ is a damping coefficient and the parameter $v_0^{-1}\sim k_0\tau_0$, $v^2(t)\ll v_0^2$. The contribution of impurities to this collision term has been neglected. In view of expressions (5) and (6), we find the correlation function of the thermal fluctuation forces

$$\langle\zeta(t),\zeta(t')\rangle=(\gamma T/\tau_0)\exp(-|t-t'|/\tau_0) . \tag{8}$$

The interaction of the particle with the dissipative system results in the usual diffusion with the displacement correlation function

$$\langle r_i(t),r_j(t')\rangle=2D\delta_{ij}\min(t,t') , \tag{9}$$

where $D=T/\gamma$ is a thermal diffusion coefficient ($m=1$).

The impurity scattering of the Brownian particle is described by the random potential $Q(r)$. For the sake of simplicity we restrict ourselves to a one-dimensional case ($d=1$) and assume that $Q(x)$ is equal to the superposition of chaotically distributed potential wells $Q_{\text{imp}}(x)$ [7]. If, for example, $Q_{\text{imp}}(x)=Q_0\exp(-2x^2/r_c^2)$, then the correlation function of the random environment can be written in the form

$$M_k=\langle Q_k,Q_{-k}\rangle=M_0\exp(-k^2r_c^2/4) ,$$
$$M_0=(\pi/2)n_ir_c^2Q_0^2 , \tag{10}$$

where n_i is the impurity concentration and r_c is the finite correlation length of the disordered medium.

In the frames of a weak disorder approximation, the parameter

$$\lambda^2=n_ir_cQ_o^2/T^2\ll 1 .$$

Therefore, the relative contribution of impurity scattering to the particle relaxation is also negligible: $\gamma_{\text{imp}}/\gamma\sim\lambda^2(\gamma r_c/v_T)\ll 1$. It should be mentioned that the time dependence of random forces $\xi_j(t)$ is a consequence of the time evolution of Brownian particle displacements $r_j(t)$. We may suppose that this evolution is mainly connected with the usual thermal diffusion, described by (9). Taking into account terms of order λ^2, we obtain from (5) the correlation function of random forces

$$\langle\xi(t),\xi(t')\rangle=L^{-d}\sum k^2M_ke^{-Dk^2(t-t')}e^{ikV_d(t-t')} . \tag{11}$$

Here $V_d=E/\gamma$ is a drift velocity of the Brownian particle. The components of random sources having the wave vectors $k<(D\tau)^{-1/2}$ give the dominant contribution to the random force correlator (11). As this contribution is proportional k^{d+2}, the equilibrium velocity correlation function ($V=0$) demonstrates a power law asymptotic behavior at $L=\infty$ ($\tau\gg 1/\gamma\gg\tau_0$),

$$\langle v(t+\tau),v(t)\rangle\sim\tau^{-(d+2)/2} . \tag{12}$$

However, the corrections to the diffusion coefficient due to impurity scattering are small: $D_{\text{imp}}/D\sim\lambda^2(\gamma r_c/v_T)\ll 1$.

If the sample size L is finite, then the correlator (11) exponentially decreases at times $\tau_L=L^2/4\pi^2D$. With al-

lowance for terms order λ^4 we also come to the conclusion that the equilibrium velocity correlation function $\langle v(t),v(t')\rangle$ vanishes at large times. As a result, the second-order cumulant function of particle kinetic energy fluctuations [or fluctuations of the velocity square $v^2(t)$] equal the fourth-order velocity cumulant function in the limit $t-t' \gg 1/\gamma$. But this cumulant function is determined by the corresponding cumulant function of random forces $\xi(t)$,

$$K(t,t')=\langle v^2(t),v^2(t')\rangle = (1/\gamma^4)\langle \xi(t),\xi(t),\xi(t'),\xi(t')\rangle \ .$$

Here we suppose that $\gamma > v_T/r_c$. Notice that by virtue of the nonlinear coordinate dependence, the random force $\xi(t)$ is non-Gaussian, even though the random potential variable Q_k itself is Gaussian. To calculate this cumulant function, we have to use the Furutsu-Novikov theorem [6]. The form (5) of the random source allows us to except pairings between components Q_k and variables $\exp(ikx(t))$ belonging to the same random force $\xi(t)$. The maximum contribution to $K(t,t')$ gives pairings of components Q_k and Q_k, connected with random forces $\xi(t)$ and $\xi(t')$, respectively. As a result, we get the expression for the cumulant function $K(t,t')$,

$$K(t,t')=2\gamma^{-4}L^{-2}\sum_k \sum_q k^2q^2M_kM_q \times\langle \exp\{i(k+q) \times[x(t)-x(t')]\}\rangle \ . \tag{13}$$

In the quenched disorder model the time evolution of $K(t,t')$ is completely determined by the time dependence of the particle displacement $x(t)$. Therefore, the term in (13) with $q=-k$ does not vanish at large times. Taking into consideration the particle diffusion at the relaxation time ($v_T/r_c \sim 1/\gamma$) and using the standard rule of a transition from a sum to an integral,

$$(1/L)^d\sum_k \to \int d^dk/(2\pi)^d \ ,$$

we obtain a more exact expression of the kinetic energy correlation function ($d=1$) in the long time limit ($t \gg t'$),

$$K(t,t')=(1/\pi L)\int dk \frac{k^4M_0^2\exp(-k^2r_c^2/2)}{(\gamma^2+Tk^2)^2} \ . \tag{14}$$

Note that the terms of the higher powers of λ^2 give small corrections to $K(t,t')$. In the lower temperature domain ($T<T_c=m\gamma^2r_c^2$) the level of persistent correlations almost does not depend on the temperature T of the dissipative environment. At the same time the value of these correlations is proportional T^{-2} at higher temperatures ($T \gg T_c$),

$$K(t,t')=(\pi/2)^{3/2}\frac{n_i^2Q_0^4}{2L\gamma^4r_c}\begin{cases}6, & T<T_c\\ (m\gamma^2r_c^2/T)^2, & T\gg T_c.\end{cases} \tag{15}$$

Due to the nonlinearity of collision term $G[v]$ (7), the long time tail of the kinetic energy correlation function (15) will be manifested in the nonequilibrium second-order cumulant function of the particle velocity ($t-t' \to \infty$),

$$\langle v(t),v(t')\rangle = 9\langle v^2(t),v^2(t')\rangle(V_d^2/v_0^4) = 9(\pi/2)^{3/2}\frac{n_i^2Q_0^4V_d^2}{2L\gamma^4r_cv_0^4} \times\begin{cases}6, & T<T_c\\ (T_c/T)^2, & T\gg T_c,\end{cases} \tag{16}$$

where $V_d=E/\gamma$ is the drift velocity. Thus, in a steady state both the fourth-order velocity cumulant $\langle v^2(t),v^2(t')\rangle$ and the second moment $\langle v(t),v(t')\rangle$ survive at large time $t-t' \to \infty$.

It is important to to note that the anomalous behavior of the kinetic energy correlation function (15) leads to the violation of the Sloutsky condition [8],

$$\lim_{\theta\to\infty}(1/\theta)\int_0^\theta \langle v^2(t),v^2(t+\tau)\rangle d\tau = 0 \ ,$$

that is, to the second-order nonergodicity of velocity fluctuations of the Brownian particle in a finite-size disordered medium. As a result, the relative root-mean-square error of the measurement of the mean square $\langle v^2\rangle$ by time averaging procedure ($\theta \to \infty$),

$$(1/\theta)\int_{t-\theta}^{\theta} v^2(\tau)d\tau \ ,$$

tends to the nonzero value $\sigma=K(t,t')/v_T^4(t-t' \to \infty)$, which is proportional to the relative level of kinetic energy residual correlations. In the presence of an external bias the fluctuations of the particle velocity $v(t)$ have the nonergodic character, the mean drift velocity $\langle v\rangle$ being an unmeasurable quantity.

When the damping coefficient $\gamma \cong 10^{12}$ s^{-1}, $m \cong 10^{-27}$ g, the correlation length $r_c \cong 7\times10^{-6}$ cm, $n_ir_c \cong 10$, the maximum value of the impurity potential $Q_0 \cong 30$ K, and the thermal bath temperature $T \cong 300$ K, then the parameter $\lambda^2 \cong 0.1 \ll 1$, $\gamma r_c \cong v_T$, $T \cong T_c$, and the relative level of kinetic energy residual correlations $\sigma=K(t,t')/v_T^4 \cong 6(r_c/L)$. For a micron-size ring ($L \cong 10^{-4}$ cm) the persistent fluctuations account for about 7% of the thermal noise. The relative level of nonequilibrium residual correlations of Brownian particle velocity is

$$\langle v(t),v(t')\rangle/v_T^2 \cong 2.4\times10^{-3}(r_c/L) \cong 1.6\times10^{-4} \ ,$$

if the maximum wave vector of the dissipative environment $k_0 \cong 10^7$ cm^{-1}, the thermostat correlation time $\tau_0 \cong 10^{-14}$ s, the electric field strength $E \cong 100$ V/cm, and the drift velocity $V_d=eE/m\gamma \cong 2\times10^5$ cm/s. Note that the value of the correlation length $r_c \cong 3v_T/\gamma$ is optimum for an experimental observation of this effect.

Finally, we note that the contribution of impurity scattering to the low-frequency spectrum of nonequilibrium velocity fluctuations may be prominent if we go beyond the scope of the quenched disorder model and take into account the slow walk of impurities or defects. As in the case with $1/f$ noise, the spectrum of residual

correlations is proportional to the square of the external bias E^2 and to the inverse volume of the sample L^{-d}. Therefore, we suppose that the above-mentioned generalization of our results will allow us to clarify the nature of experimentally observed excess $1/f$ noise accompanying transport processes in a finite-size inhomogeneous system.

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